6. MOTION: DISPLACEMENT, VELOCITY AND ACCELERATIONS

OBJECTIVES

to learn through experience, experimentation and theoretical developments
the nature and mathematical (geometrical and analytical) descriptions of
- linear motion
- rotational motion
- the conditional relationships between them.

You will be able to
- describe both linear and rotational motion geometrically and analytically.
- solve a range of 1 and 2 dimensional linear motion problems
- solve a range of rotational motion problems
- solve problems involving combined linear and rotational motion

Linear Displacement
Rotational Displacement
The connection - rolling without slipping.

THE EVENTS

1. Carefully observe moving vehicles - bicycles, cars, trucks, buses and whatever else. Observe the translational (linear) motion as the vehicle travels down the street or road. Observe the rotating wheels on moving vehicles. Note any relationship between the linear motion of the vehicles and the rotation of the wheels.

2. Carefully study the figure below. Study the positions, velocities and accelerations versus time in each situation. As you gain more knowledge and understanding, explain to yourself, and to others, even if in your imagination, what it is all about.

6.1 Definitions and Relationships for Translational (Linear) Motion

6.1.1 Definitions for Translational (Linear) Motion

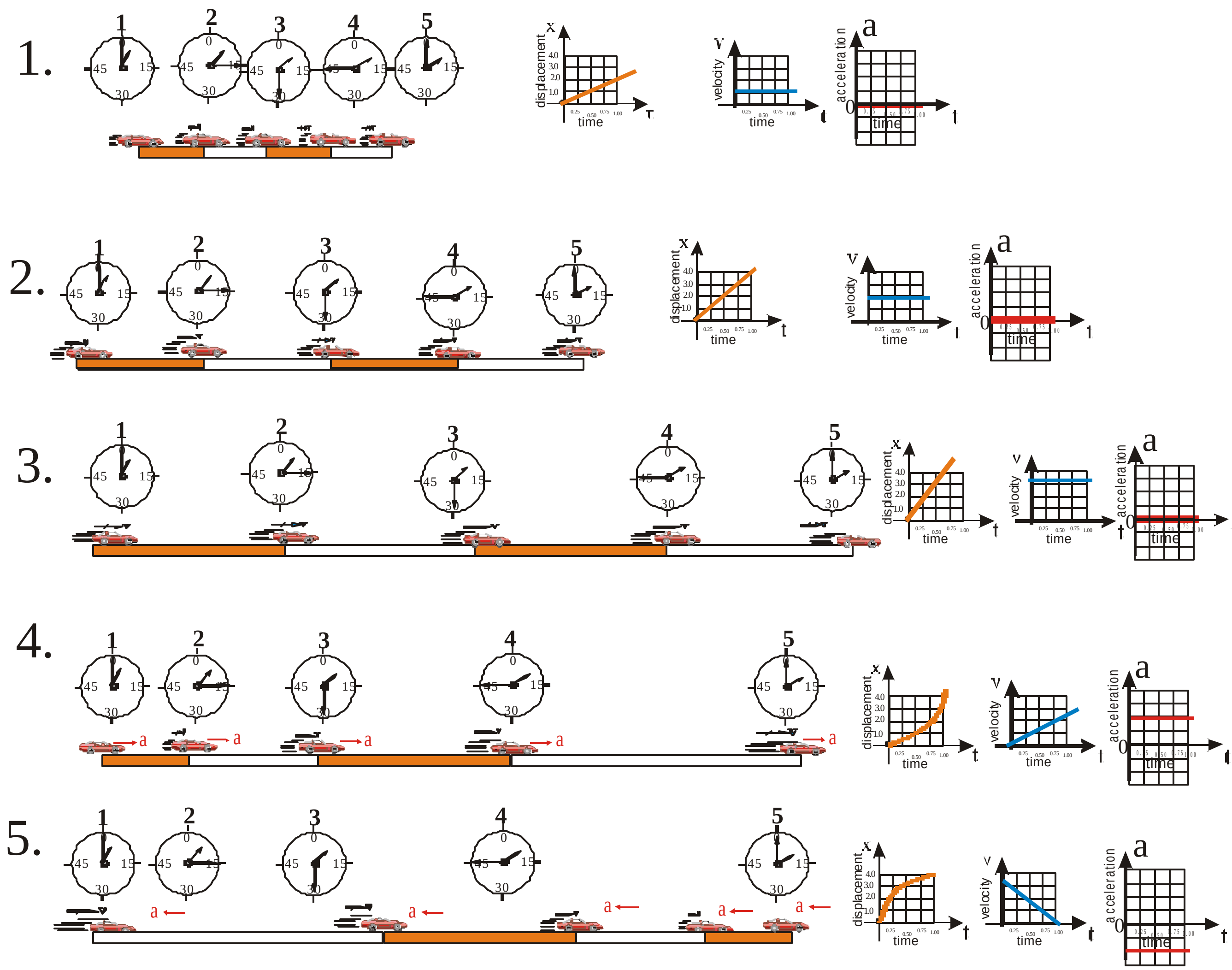


Figure 1

Pictorial Time Lapse Examples of 1 Dimensional Translational (Linear) Motion
Each picture above shows a car traveling along a straight path. Its position is shown at equal time intervals with a clock above it displaying the elapsing time. Graphs of the linear displacement versus time (x vs t), velocity versus time (v vs t) and acceleration versus time (a vs t) are shown for each situation. In 1 to 3 the car travels with three different constant velocities, slowest in 1 and fastest in 3. In 4 the car starts from rest and travels at a constant positive acceleration. In 5 the car travels a constant negative acceleration.

Note the graphs, and the velocity and acceleration vectors.

6.1.2 Definitions and relationships for Linear Motion x,v,a vs t

A. Displacement x [L] is a change in position

displacement divided by time gives velocity

B. Velocity v [L/T] is the time rate of change in displacement. Definition:

$v = \Delta d/\Delta t$ slope of d vs t graph.

= derivative of d with respect to t when the function d(t) exists and is differentiable.

velocity divided by time gives acceleration

C. Acceleration a [L/T^2] is the time rate of change in velocity.

Definition:

$a = \Delta v/\Delta t$ slope of v vs t graph.

= derivative of v with respect to t when the function v(t) exists and is differentiable.

. **<u>Velocity -> Displacement</u>**

v = Δ**x**/Δt, cross multiply by Δt gives

Δx = v Δt , v Δt is an area element under the v vs t graph. So, the total displacement Δ**x** is the area under the v vs t graph.

velocity for a time interval causes an change in displacement

<u>Acceleration -> Velocity</u>

a = Δv/) t, cross multiply by Δt gives

Δv = a Δt , a Δt is an area element under the a vs t graph. So, the total change in velocity, Δ**v**, is the area under the a vs t graph.

acceleration for a time interval causes an change in velocity

a acceleration

6.1.3 Average and Instantaneous velocity and acceleration

Average values are from the differences of observations over arbitrary time intervals, that is, the time intervals can be a long or as short as you please. So, the average velocity and acceleration are defined by the following differences:

$$v_{average} = v_{avg} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{x(t_2) - x(t_1)}{t_2 - t_1} = \frac{\Delta x}{\Delta t} \quad , \quad \text{and} \quad a_{average} = a_{avg} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{v(t_2) - v(t_1)}{t_2 - t_1} = \frac{\Delta v}{\Delta t} \quad ,$$

where Δt is an arbitrary time interval.

Instantaneous values are obtained by taking observations over increasingly shorter and shorter time intervals, until the interval approaches zero. So,

$$v_{instantaneous} = v(t) = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

and

$$a_{instantaneous} = a(t) = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} \quad .$$

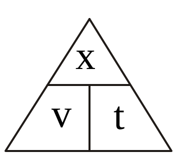
Hence, the instantaneous values are derivatives.

Our eyes run at a frame rate of about 10 per second. That is, we take about 10 pictures per second. Therefore we do not physically observe instantaneous velocity and acceleration, but only approximate it. Slow moving objects appear a sharp, well defined images because their boundaries are defined by single cones in the retina. On the other hand, very fast moving objects appear as a blur because their boundaries are spread over several cones in the retina. Thus they are not well defined .

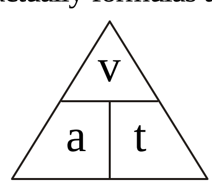
In all that follows we will be referring to instantaneous values unless specially stated.

6.1.4 Memory Aids

The triangular figures below are examples of a class of memory aids. You will encounter many examples throughout this document. They are not so much intended for remembering exact formulas, but to help remembering the basic dimensional relationships between the variables. There are, however, special cases when they are the exact formulas. But, usually the actually formulas are more complex



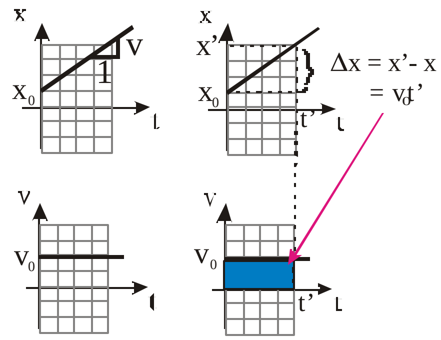
$x = vt$,
$v = x/t$,
$t = x/v$



$v = at$,
$a = v/t$,
$t = v/a$

6.2 Special Case of Constant Velocity

This is the simplest case. The figure on the right follows from the right side of page 34, above - "Velocity -> Displacement" .



Exercise:

What is x(t)?, x as a function of t?

Write the equation.

6.2.1. 1 - Dimensional Case

Here there is only one spatial dimensional, say x. or y, horizontal or vertical. North or East, etc - one or the other, but not both. Taking x we get the relations are

$$\Delta x = x(t) - x_0 = v\,t, \quad \text{so} \quad x(t) = x_0 - v\,t.$$

Review Figure 1.

6.2.2 2 - Dimensional Case

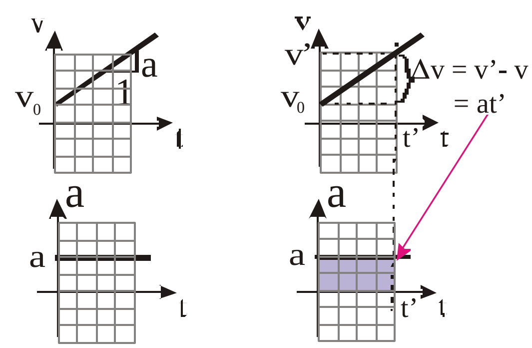
Here there are two spatial dimensions, say x and y, or vertical and horizontal, front and left, North and East, etc. - always a pair. Taking x and y we get the relationships

$$\Delta x = x(t) - x_0 = v_x\,t \quad \text{and} \quad \Delta y = y(t) - y_0 = v_y\,t\,,$$

where v_x and v_y are the x and y components of the constant velocity, respectively. This can be reduced or transformed to motion of constant velocity along a straight line, hence to a 1-dimensional motion.

6.3 Special Case of Constant Acceleration

We look at the special case of constant acceleration. The figure on the right follows from the right side of Sec 6.1.1, above - "Acceleration -> Velocity" .



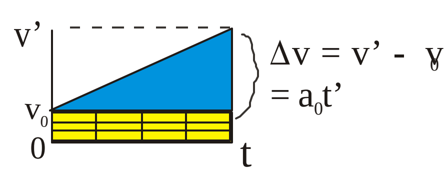
Here the basic relation is

$$\Delta v = v(t) - v_0 = at,$$

where t is the elapsed time from t=0.

6.3.1. 1 - Dimensional Case

Let's take the one spatial direction to be x .From the figure and its reference in Sec 6.1.1, above - "Velocity -> Displacement" we have the change in displacement Δx is



Δx = area under v vs t curve

= area of triangle + area of rectangle

= base x height+ ½ base x height

$= \quad v_0\,t \quad + \quad \frac{1}{2}\,(v' - v_0)\,t$

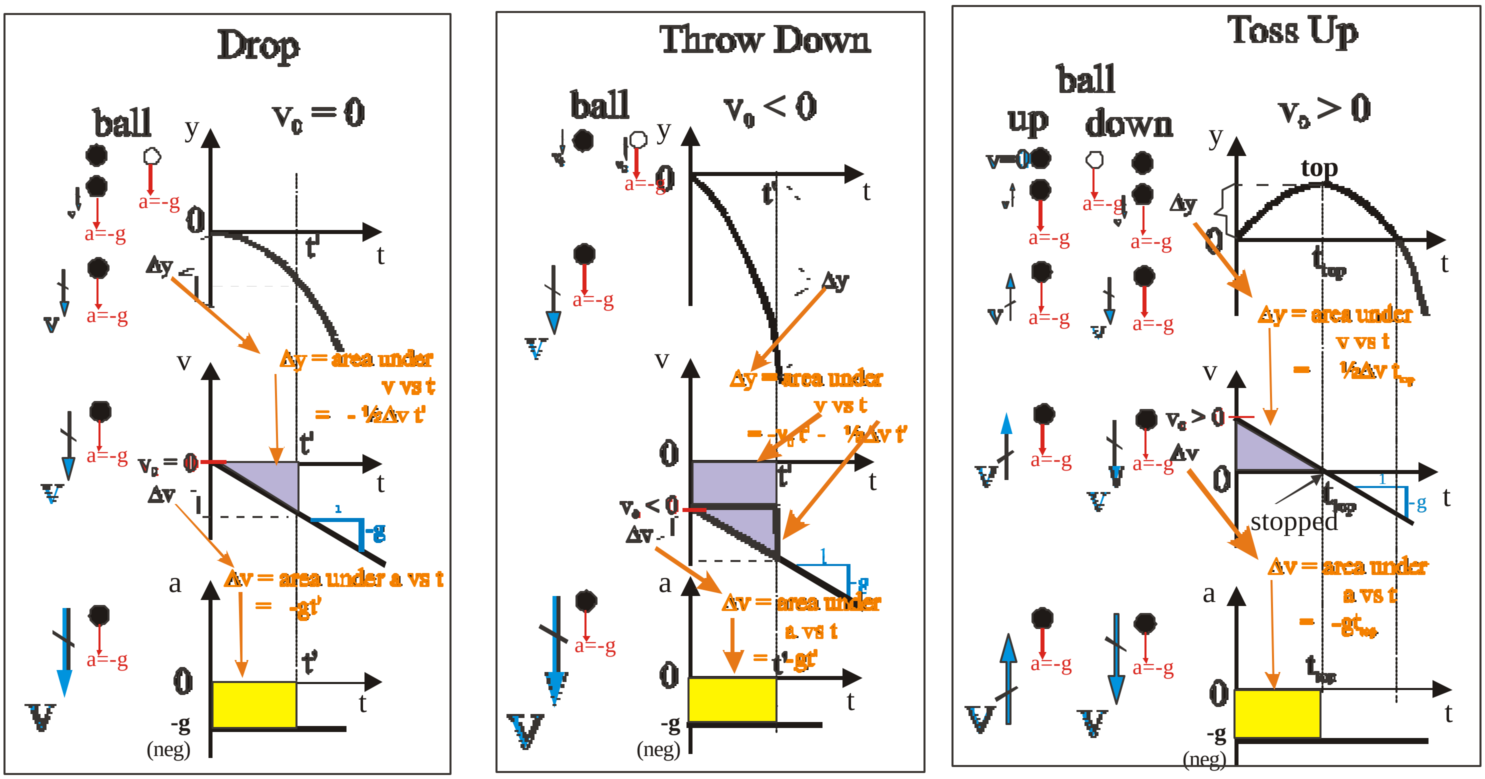
$= v_0\,t + \frac{1}{2}a\,t \times t = + v_0\,t + \frac{1}{2}\,at^2\,.$

Review Figure 1.

6.4 Special Case of 1-D Constant Acceleration - Specifically Free Fall

I Dimensional Free Fall is in the vertical direction, up and down. It has three situations as illustrated in the figures below - drop ,throw downwards and throw upwards . Each shows a ball in its path and the x,v,a vs t graphs The balls are at their various positions with their velocity and acceleration vector at each position. The white ball is the initial position and condition. The x,v,a vs t graphs exhibit the areas that contribute to each of the indicated Δx and Δv .

Carefully note the velocity and acceleration vectors, and the position of each ball.



Numerical values for free fall on Earth $g=9.81\ m/s^2$

	DROP		THROW DOWN		TOSS UP	
	v0 = 0 m/s		v0 = -9.80 m/s		v0 = + 9.80 m/s	
t (sec)	v (m/s)	y (m)	v (m/s)	y (m)	v (m/s)	y (m)
0	0.00	0.00	-9.80	0.00	9.80	0.00
0.2	-1.96	-0.20	-11.76	-2.16	7.84	1.76
0.4	-3.92	-0.78	-13.72	-4.70	5.88	3.14
0.6	-5.88	-1.76	-15.68	-7.64	3.92	4.12
0.8	-7.84	-3.14	-17.64	-10.98	1.96	4.70
1	-9.80	-4.90	-19.60	-14.70	0.00	4.90
1.2	-11.76	-7.06	-21.56	-18.82	-1.96	4.70
1.4	-13.72	-9.60	-23.52	-23.32	-3.92	4.12
1.6	-15.68	-12.54	-25.48	-28.22	-5.88	3.14
1.8	-17.64	-15.88	-27.44	-33.52	-7.84	1.76
2	-19.60	-19.60	-29.40	-39.20	-9.80	0.00
2.2	-21.56	-23.72	-31.36	-45.28	-11.76	-2.16
2.4	-23.52	-28.22	-33.32	-51.74	-13.72	-4.70
2.6	-25.48	-33.12	-35.28	-58.60	-15.68	-7.64
2.8	-27.44	-38.42	-37.24	-65.86	-17.64	-10.98
3	-29.40	-44.10	-39.20	-73.50	-19.60	-14.70

Exercises:

1. Using the concepts above, verify all of the values in the table on the left.

2. Plot on graph paper the three v vs t and y vs t data in the table.

3. Using the concepts above, and that) $v = v - v_0$ and) $y = y- y_0$, show that

$$v = v_0 - gt \quad . \quad (1)$$

and

$$y = y_0 + v_0 t - ½gt^2 . \quad (2)$$

4. Throw a ball or other small object upward into the air. Follow its motion with your eyes. Explain to yourself what you see happening. Compare your observations with the above graphs for $v_0 > 0$ and the table for $v_0 = +9.8\ m/s^2$

.

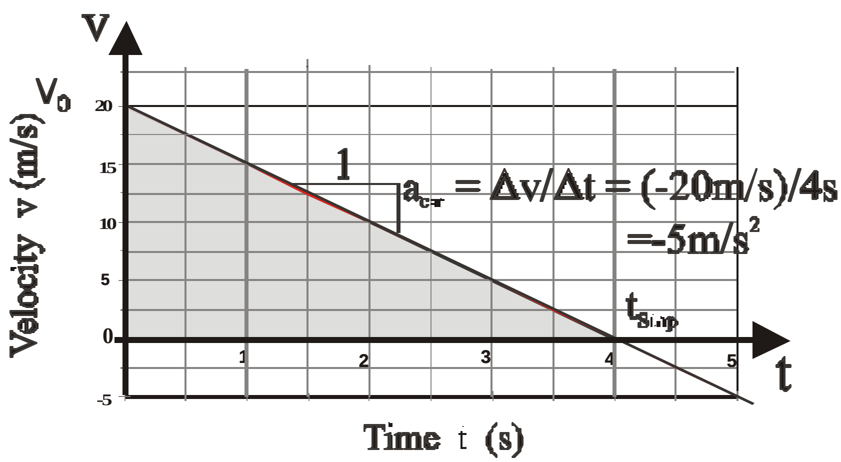
5. Drop a ball from various heights and measure the time of fall as best you can. Many digital watches, $10 and up, have a 1/100 second stopwatch feature.

6.5 Examples:

6.5.1 A Stopping Problem. A car with an initial velocity v_0 = 20 m/s puts on its breaks and comes to a stop in time t_{Stop} = 4 s while having a constant acceleration. What was its acceleration of the car, a_{car}, and the displacement of the stop, x_{Stop} ?

Overview:
First let's sketch the velocity vs time (v vs t) graph. Since the acceleration is constant, the v vs t curve is a straight line cutting the v axis at 20 m/s, the value of v_0, and the t axis at 4 s, the value of t_{Stop}. The acceleration a_{car} is the slope of the line,

Diagram:



Principle 1:

acceleration a = slope of the v vs t curve.

Details & Solution 1:

$$a_{car} = \frac{\Delta v}{\Delta t} = \frac{v_{final} - v_0}{\Delta t}$$

substitutingthenumbersgives

$$a_{car} = \frac{0 - 20\frac{m}{s}}{4s} = \frac{-20\frac{m}{s}}{4s} = -5\frac{m}{s^2}$$

Principle 2:
The displacement is the area under the v vs t curve - the shaded triangle.

Details & Solution 2:
So x_{Stop} is

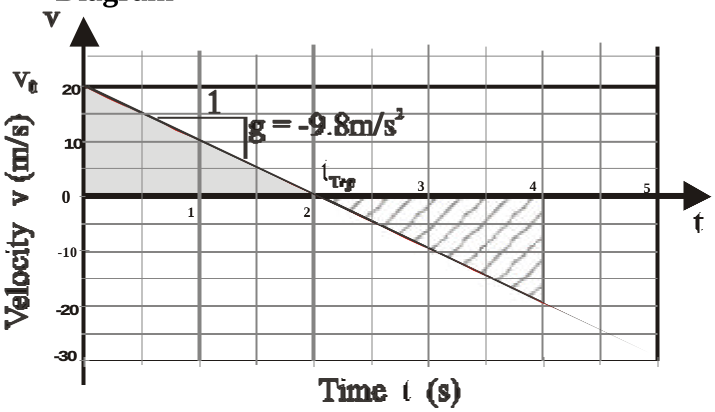
$$x_{stop} = \Delta x = areaundervvstcurve = \frac{1}{2} base * height = \frac{1}{2} v_0 t_{stop} = \frac{1}{2} * 4s * 20\frac{m}{s} = 40m$$

6.5.2 A Free Fall Problem. A ball is thrown upwards with an initial velocity v_0 = 20 m/s up. How long does it take to get to the top, t_{atTop}, ? How far,) y, does it rise? How long does it take to come back to it original height, t_{Back}?

Overview

For free fall the acceleration is constant at a value of g = 9.8m/s^2 down on earth. Taking up as positive, makes downward g = - 9.8m/s^2 . So the v vs t curve is a straight line with slope -9.8 m/s^2 that cuts the v axis at v_0 = 20 m/s (+ because up has been **chosen** as positive).

Diagram



Principle 1: acceleration a = slope of the v vs t curve,

Details & Solution 1:

$$a=\frac{\Delta v}{\Delta t}=\frac{v_{final}-v_0}{t_{final}-0}=\frac{0-20\frac{m}{s}}{t_{atTop}-0}=g=-9.8\frac{m}{s^2}$$

Principle 2:

t_{atTop} follows from the definition of acceleration as the slope of the v vs t curve.

t_{atTop} =) t =) v/a

Details &Solution 2:

$$t_{atTop}=\frac{(v_{final}-v_0)}{g}=\frac{0-20\frac{m}{s}}{-9.8\frac{m}{s^2}}=\frac{-20\frac{m}{s}}{-9.8\frac{m}{s^2}}=2.04\frac{m}{m}\frac{s^2}{s}=2.04s$$

Principle 3:

The displacement is the area under the v vs t curve. This is the area of the light shaded triangle on the left .

Details & Solution 3:

$$\Delta y = areaundervvstcurve=\frac{1}{2}base*height=\frac{1}{2}v_0\,t_{stop}=\frac{1}{2}*2.04s*20\frac{m}{s}=20.4m$$

Finally, The falling displacement is the area of the darker shaded triangle on the right and below the t-axis. But, the ball must fall the same displacement that it rose. So, the two triangles must have the same area. Both make the same angle with the t-axis. Both have corresponding sides that are perpendicular to the t-axis. The sum of the angles for all plane triangles is 180^o. So, the two triangles are similar because their corresponding angles are equal. They are congruent because they have the same area. Hence they have corresponding side equal. Thus it takes the same amount of time to fall back as it did to rise - $t_{atTop} = t_{Back}$.

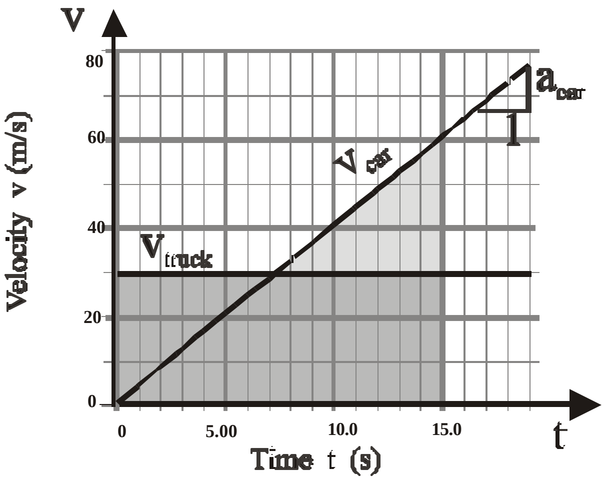
6.5.3 The Pursuit Problem. A truck speeds along the highway with a constant velocity v_T = 30m/s. As it passes a police car, the police car starts to chase the truck. The police car starts form rest and has a constant acceleration a_c = 4 m/s^2. In terms of v_T and a_{car}, find the time , t_c, distance traveled, x_c, and the speed of the police car, v_{car}, at the instant that the police car catches up with the truck.

Identifying the a crucial, critical aspect of the problem

Catchup means that both vehicles are at the same x position x_c at the same time $t = t_c$.

$$x_c = x_{Truck} = x_{car} \quad \text{when } t = t_c , \qquad (1)$$

Diagram



Principles:

1. The change in displacement, x, of each vehicle is the area under their respective velocity versus time (v vs t) curves.
2. The acceleration is the slope of the v vs t curve.

Details :

Both vehicles have constant acceleration, zero for the truck and a_{car} for the car. So, the velocity vs time (v vs t) curve of each is a straight line; a straight line with zero slope and with v-intercept v_T = 30 m/s for the truck , and a straight line with positive slope a_{car} and v-intercept 0 for the car. They are shown in the diagram. We assumed that the police car starts from rest the instant the truck passes by it. So, the truck gets no "head start". That is, they both have the same initial position x_0 =0. Hence the displacements at the catchup time t_c are the areas under the respective v vs t curves . The area under each is

$$x_{Truck} = \text{ area of rectangle } = \text{height x base } = v_T t_c \qquad (2)$$

and

$$x_{car} = \text{area of triangle} = \tfrac{1}{2} \text{ height x base } = \tfrac{1}{2} v_{car} t_c = \tfrac{1}{2}(a_{car} t_c) t_c = \tfrac{1}{2} a_{car} t_c^2 . \qquad (3)$$

Using Eq(2) and (3) in Eq(1) gives us

$$v_T t_c = \tfrac{1}{2} a_{car} t_c^2 \qquad (4)$$

Canceling a t_c on each side and solve to t_c gives

$$t_c = 2v_T / a_{car} \quad . \qquad (5)$$

The numbers give

$$= 2x(30m/s)/(4m/s^2) = 15 \text{ s} \quad .$$

To find x_c , it view of Eq. (1), it is easier to substitute Eq(5) into Eq(2) to get

$$x_c = v_T t_c = v_T \text{ x } 2v_T / a_{car} = 2v_T^2 / a_{car} \quad . \qquad (6)$$

The numbers give

$$x_c = 2 \text{ x } (30m/s)^2 /(4m/s^2) = 2 \text{ x } 900 \, (m^2/s^2) /(4m/s^2) = (2x900/4)m = 450 \text{ m} .$$

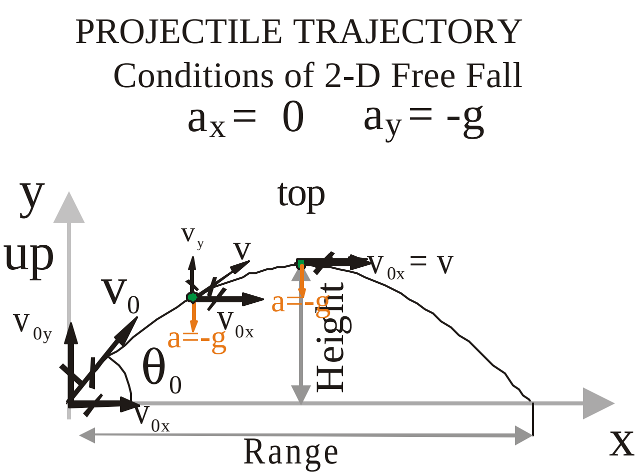
The velocity of the car at catch up is

$$v_{car\ at\ catchup} = v_{car\ 0} + a_{car}t_c = 0 + a_{car}(2v_T/a_{car}) = 2v_T \quad . \qquad (7)$$

The conclusion is that under the conditions stated in the first three sentences **the speed of the pursuer**, the police car here, is **always twice the speed of the pursued at catchup**, the truck here.

6.6 PROJECTILE TRAJECTORIES - 2D Free Fall

2 Dimensional Free Fall is the combination of 1-Dimensional Free Fall in the vertical direction (See Section 6.4) and constant velocity in the horizontal direction (See Section 6.2) . That is $a_x = 0$ and, choosing up as plus, $a_y = -g$ (= -9.8 m/s^2 on earth). The picture below shows the general situation. The body starts at $x=x_0$ (= 0 here) and $y = y_0$ (=0 also) with a velocity v_0 making an angle θ_0 with the horizontal . In general , y_0 and x_0 can have any arbitrarily chosen values. Its vertical velocity, v_y , slows as it rises. Stops at the top (Height) and starts to fall. Simultaneously, it is traveling in the horizontal direction with a constant velocity $v_x = v_{0x}$.When the body returns to the vertical position y_0, it has traveled a horizontal displacement R called the Range. A body that travels this type of path, simultaneous horizontal and vertical motion, is called a projectile. This example is the ideal situation in which friction and wind have no influence.



From the above picture it immediately follows that

$$v_{0x} = v_0 \cos\theta_0 \quad \text{and} \quad v_{0y} = v_0 \sin\theta_0 \quad . \qquad (1)$$

The horizontal motion, the x direction, is as in Section 6.2.1 and the vertical motion, the y direction, is as in Section 6.3.1 and 6.4 Eqs. (1) and (2) . These yield

$$v_x(t) = v_{0x} = v_0\cos\theta_0 \qquad v_y(t) = v_{0y} - gt = v_0\sin\theta_0 - gt \qquad (2)$$

$$x(t) = v_{0x}t = v_o\cos\theta_0 \qquad y(t) = y_0 + v_{0y}t - ½gt^2 = y_0 + v_0\sin\theta_0 t - ½gt^2 \qquad (3)$$

for the velocities and displacements, respectively.

First, discover the relation between x and y, the spacial curve y(x) for this situation. Here, there is no interest in time, t. So, eliminating it.

Eq. (1) immediately yields $t = (v_0\sin\theta_0 - v)/g$ (4)

Substituting Eq. (3) into Eq. (2) and doing some algebra yields

$$y(x) = y_0 + x \tan\theta_0 \; - gx^2/(2[v_0\cos\theta_0]^2) \; . \qquad (5)$$

The range R is defined as the distance the projectile travels to return to its original height,

$$y(R) = R\tan\theta_0 \; - gR^2/(2[v_0\cos\theta_0]^2) = R\sin\theta_0/R\cos\theta_0 \; - gR^2/(2[v_0\cos\theta_0]^2) = 0 \; .$$

Solving for R yields

$$R = 2\, v_0^2 \; \cos\theta_0 \sin\theta_0/g = \; v_0^2 \; \sin 2\theta_0/g \; . \qquad (6)$$

after using the trig identity $2 \cos\theta_0 \sin\theta_0/g = \sin 2\theta_0$.

Now let's get the expression for the height, H, in terms of v_0 , θ_0 . The easiest way to do this is to use the y-axis part Eqs. (2) and (3) as follows. The velocity is v(t) = 0 when y = H. So, find the time t_{top} that reach the top by solving from Eq. (2)

$$v_y(t) = 0 = v_0\sin\theta_0 - gt_{top} \quad .$$

This yields

$$t_{top} = v_0\sin\theta_0 /g \quad . \qquad (7)$$

Using $y(t_{top}) = H$ in Eq. (3) gives

$$y(t_{top}) = H \; = \; y_0 + \; v_0\sin\theta_0 t_{top} - ½g \; t_{top}^2$$

which gives after substituting Eq. (7)

$$H = ½\, (v_0 \sin\theta_0)^2 /g \quad . \qquad (8)$$

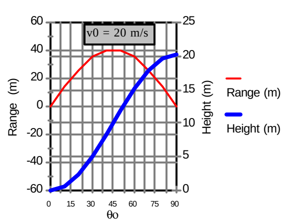
A more difficult method knowing that at the top the curve y(x) is flat, its derivative is zero . We get from Eq. (5)

$$dy(x)/dx = 0 = d/dx[\; y_0 + x \tan\theta_0 \; - gx^2/(2[v_0\cos\theta_0]^2)] \; .$$

Solve for $x = x_{top}$, then substitute expression for x_{top} into Eq. (5) to get H.

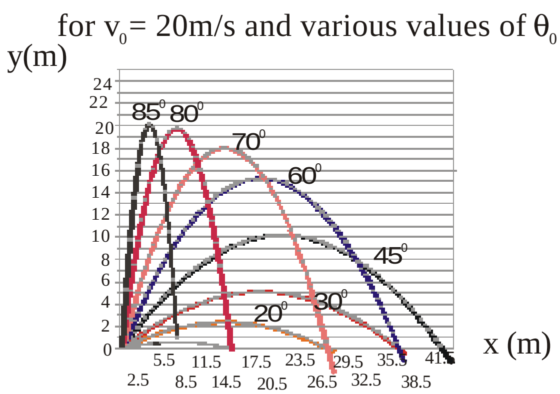
6.6.1 Exercise: By direct calculation verify all of the values of y in tables below.

Graphs - Range and Height for v_0 =20 m/s vs θ_0



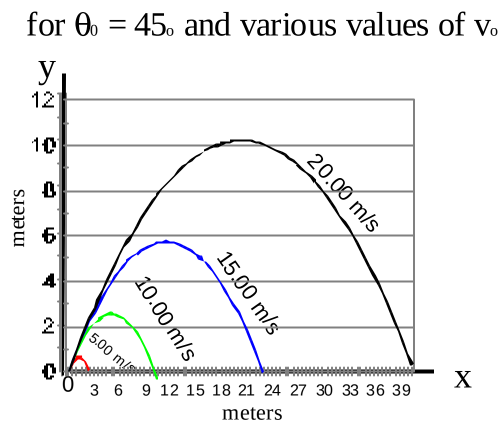
0	Range (m)	Height (m
0	0	0
10	13.96	0.62
20	26.24	2.39
30	35.35	5.10
40	40.20	8.43
50	40.20	11.98
60	35.35	15.31
70	26.24	18.02
80	13.96	19.79
90	0.00	20.41

Graphs - v_0 =20 m/s at various values of θ_0



v0 =20m/s		0			
0 x	45.0 deg y	30.0 deg y	60.0 deg y	20.0 deg y	70.0 deg y
0	0	0	0	0	0
4	3.61	2.05	6.14	1.23	9.31
8	6.43	3.57	10.72	2.02	15.28
12	8.47	4.58	13.73	2.37	17.89
16	9.73	5.06	15.17	2.27	17.15
20	10.20	5.01	15.04	1.73	13.06
24	9.89	4.45	13.35	0.74	5.62
28	8.79	3.36	10.08	-0.69	-5.17
32	6.91	1.75	5.25	-2.56	-19.31
36	4.25	-0.38	-1.15	-4.88	-36.81

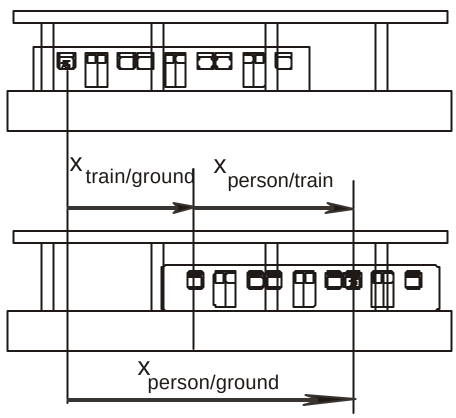
Graphs - Trajectories θ_0 = 45° for various values of v_0



v0 x	5.00 m/s y	10.00 m/s y	15.00 m/s y	20.00 m/s y
0	0.00	0.00	0.00	0.00
3	-0.53	2.12	2.61	2.78
6	-8.11	2.47	4.43	5.12
9	-22.75	1.06	5.47	7.02
12	-44.45	-2.11	5.73	8.47
15	-73.20	-7.05	5.20	9.49
18	-109.01	-13.75	3.89	10.06
21	-151.87	-22.22	1.79	10.20
24	-201.79	-32.45	-1.09	9.89
27	-258.77	-44.44	-4.75	9.14
30	-322.80	-58.20	-9.20	7.95
33	-393.89	-73.72	-14.43	6.32
36	-472.03	-91.01	-20.45	4.25
39	-557.23	-110.06	-27.25	1.74

6.7 Relative Motion

Relative motion is quite simple. The person in the train example here is the model for this classic of problems in one-dimension. Here the person travels inside the train. The person's displacement relative to the train we name $\mathbf{x}_{person/train}$. Simultaneously, the train travels in the station on the ground. The displacement of the train relative to the ground we name $\mathbf{x}_{train/ground}$. As seen from the station, that is from the ground, the person travels the displacement relative to the ground, $\mathbf{x}_{person/ground}$. It is the combined displacements of $\mathbf{x}_{train/ground}$ and $\mathbf{x}_{person/train}$. Hence the result



$$\mathbf{x}_{person/ground} = \mathbf{x}_{train/ground} + \mathbf{x}_{person/train}$$

Let's assume that these motions are occurring simultaneously. Then dividing through by the common time of the travel, Δt, yields the velocity relationship

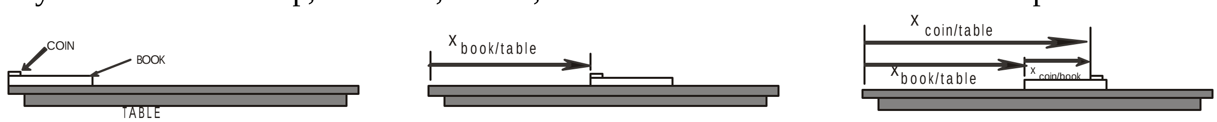
$$\mathbf{v}_{person/ground} = \mathbf{v}_{train/ground} + \mathbf{v}_{person/train}$$

Finally, since the acceleration **a** = Δ**v**/Δt, or **v**/Δt here, then dividing through by the common time of the travel, Δt, yields the acceleration relationship

$$\mathbf{a}_{person/ground} = \mathbf{a}_{train/ground} + \mathbf{a}_{person/train}.$$

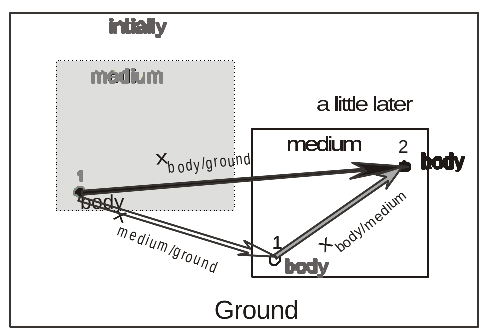
These are general vector equations that apply in 1, 2, and 3 dimensional motions.

6.7.1 Exercise: Here is a 1-D example that you can do at home, or any place, for that matter. All you need is a table top, or a floor, a book, and a can or a coin. Follow the three steps shown below.



START MOVE THE BOOK RELATIVE TO THE TABLE MOVE THE COIN RELATIVE TO THE BOOK

Now let's look at 2-Dimensional example. In the figure a body is on a medium, like a platform, a train, a ship or moving water. The body moves relative to the medium while the medium moves relative to the ground. The black dots show initial and final positions. The white dot marks original position on the medium, where it would be if it stayed stationary to the medium. The point 1 is a fixed position on the medium. Relative to the medium, the body moves from one position to another position. As in the previous case, for the displacements we have



$$x_{body/ground} = x_{medium/ground} + x_{body/medium}.$$

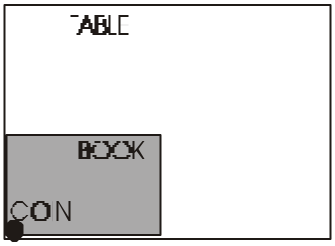
If both displacements are occurring simultaneously, then dividing through by the common time of travel, Δt, gives

$$v_{body/ground} = v_{medium/ground} + v_{body/medium},$$

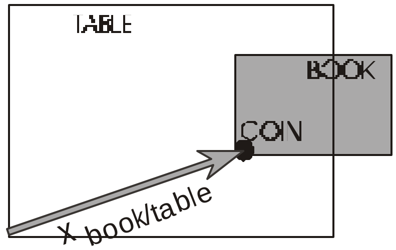
for the velocities. The view from the body relative to the ground is the vector sum of the view of the medium relative to the ground plus the view of the body relative to the medium. Do several examples. In each show a scaled diagram with lengths.

Notice the pattern of the nomenclature (the naming scheme) in the equations. It looks like a series of cancellations!

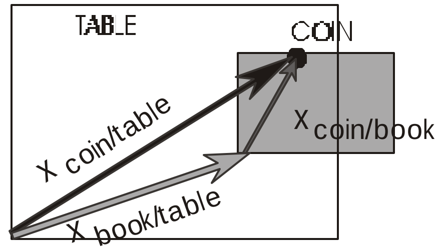
6.7.2 Exercise: Here is a 2-D example that you can do at home, or any place, for that matter. All you need is a table top, or a floor, a book, and a can or a coin. Follow the three steps shown below. Do several examples. In each show a scaled diagram with lengths and angles that you have chosen.



START WITH A COIN ON A BOOK THAT IS ON A TABLE



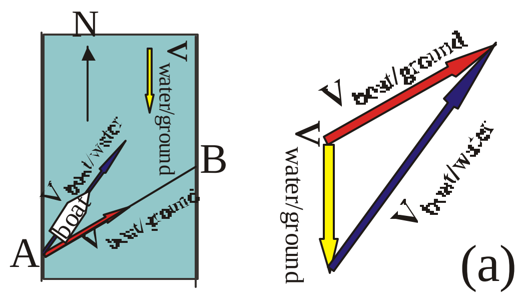
MOVE THE BOOK WHILE THE COIN STAYS FIXED ON THE BOOK



MOVE THE COIN WHILE THE BOOK STAYS FIXED ON THE TABLE

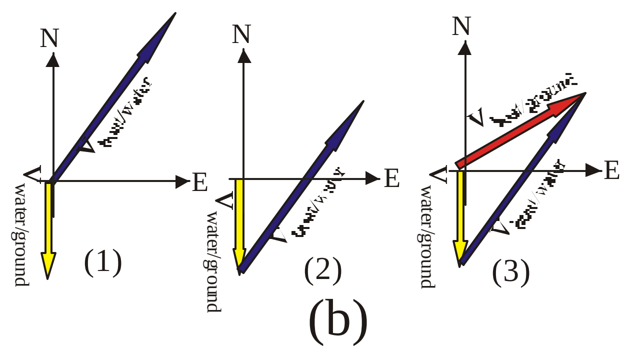
Examples: One of the most typical practical problems in relative motion is the following. A body is traveling in a moving medium, where the medium has velocity relative to ground of $v_{medium/ground}$. The body wants to get from point A to point B relative to the ground . You are given some information about the situation from which you must then find the relationship between three velocity vectors $v_{body/ground}$, $v_{medium/ground}$ and $v_{body/medium}$. Let's examine a situation and two problems that can arise from this situation.

Let's look at a boat traveling across a river to get from point A on one side to point B on the other side. The water in the river moves with a velocity relative to ground, $v_{water/ground}$. The boat velocity relative to the ground , $v_{boat/ground}$, must point in the direction from A to B. To achieve this, the boat must travel with a velocity relative to water, $v_{boat/water}$, to compensate, or correct, for the effects of the moving water on the boat path. The three velocities must satisfy the relationship



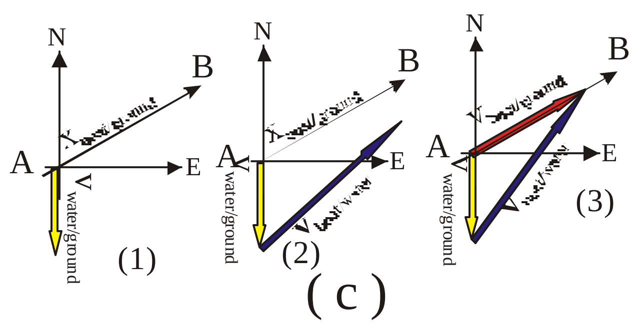
$$v_{boat/ground} = v_{water/ground} + v_{boat/water}.$$

(a) in the picture shows the situation. The three velocity vectors must be as shown. (Review Section 5.2.4.2.)



This can lead to a number of distinctly interesting problems. Let's explore two possibilities.

1. The easiest is boat wants to cross the river as shown in the diagram. The water in the river has a velocity relative to ground $v_{water/ground}$ = 5 m/s South. The boat has the velocity relative to the water = 15 m/s 60° N of E . Find $v_{boat/water}$. (b) shows how to do it. in the picture shows the situation. Here you solve for the magnitude and direction of $v_{boat/water}$. This is straight vector addition.



2. The more difficult is as follows. A boat wants to cross the river as shown in the diagram. The water in the river has a velocity relative to ground $v_{water/ground}$ = 5 m/s South. The boat has the power to maintain the velocity relative to the water $|v_{boat/water}|$ = 15 m/s. Find the direction that boat must travel relative the water to get to its destination; find the direction of $v_{boat/water}$ and the magnitude of $v_{boat/ground}$. (c) in the picture shows this situation. First, (1), draw $x_{boat/ground}$ to establish the direction of $v_{boat/ground}$. Then, (2), for the addition, put tail of $v_{boat/water}$ at the tip $v_{water/ground}$ at any angle. Finally, (3), rotate $v_{boat/water}$ to intersect $x_{boat/ground}$. This intersection point defines $v_{boat/ground}$. This is easy do graphically. The analytic solution is UGLY!

6.8 Rotational Motion

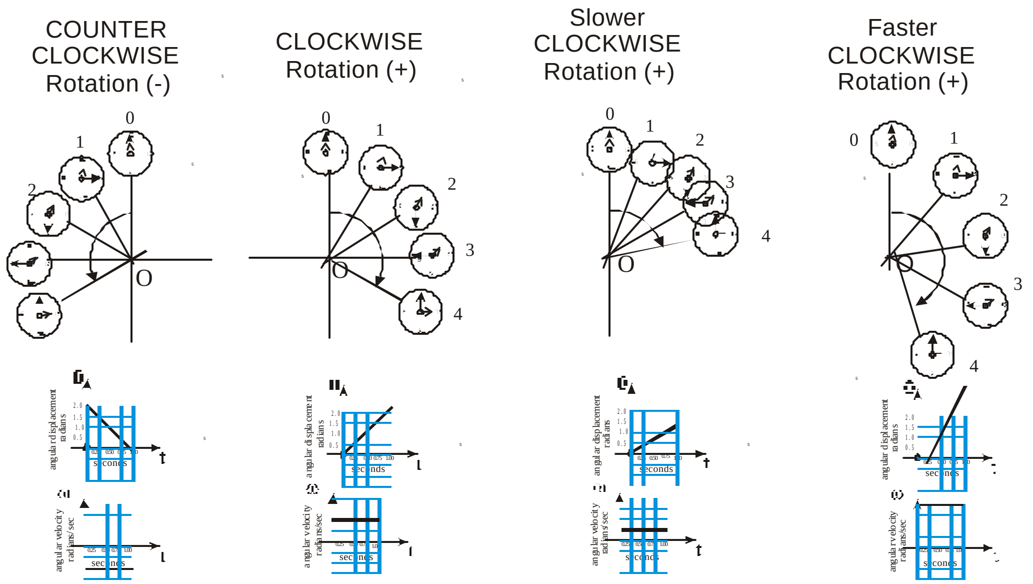


Figure 1

Pictorial Examples of Time Lapse Rotational Motion

CLOCKWISE Rotation
(chosen as + direction)

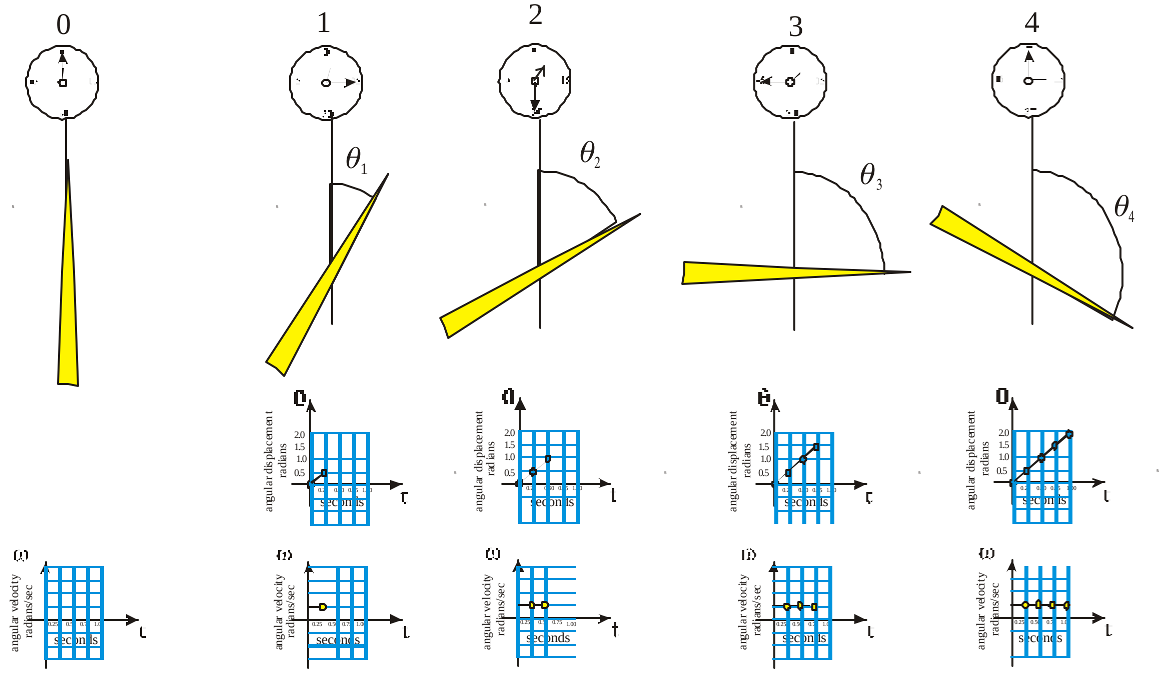
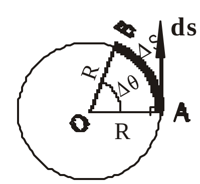


Figure 2

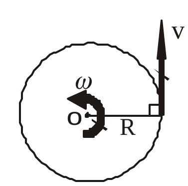
Pictorial Time Lapse Examples of Rotational Motion
Rotation is observed by a line on a body changing its orientation. Figure 1 shows a rod changing orientation at a constant rate. The clock shows the progressing time. Below each are the graphs of the angular displacement, θ , and angular velocity, ω , versus time. The graphs show each individual point as the motion progresses. Figure 2 shows rods changing orientation in various directions and at various constant rates. Clocks are attached to the rods to record the progressing change in time. Again, under each are the graphs of θ and ω vs time. The first shows clockwise rotation, while the second shows counter clockwise rotation at the same rate. The remaining depict counter clockwise rotation at a faster rate and a slower rate.

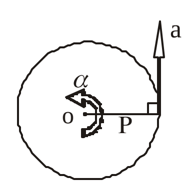


Raising awareness. **As you walk along the street carefully observe the rotating wheels on moving vehicles - cars, trucks, busses.**

6.8.1 Definitions for Rotational Motion

Rotation is observed by a line on a body changing its orientation. In the first circle below , the line from the center OA changes orientation by) 2 to become OB as the circle rotates by an angle $\Delta\theta$.





Angular Displacement q [ ] no dimension
$\Delta\theta = \omega\Delta t$ area under ω vs t curve
Units: radian (2π radian = 360 °)

Angular velocity $\boldsymbol{\omega}$ [1/T]
ω = slope of θ vs t curve = $\Delta\theta/\Delta t$
$\Delta\omega = \boldsymbol{\alpha\Delta}t$ area under α vs t curve
Units: radian/sec.

Angular acceleration $\boldsymbol{\alpha}$ [$1/T^2$]
$\boldsymbol{\alpha}$ = slope of ω vs t curve = $\Delta\omega/\Delta t$

Angular -Linear Relations

$\Delta S = R\,\Delta\theta$

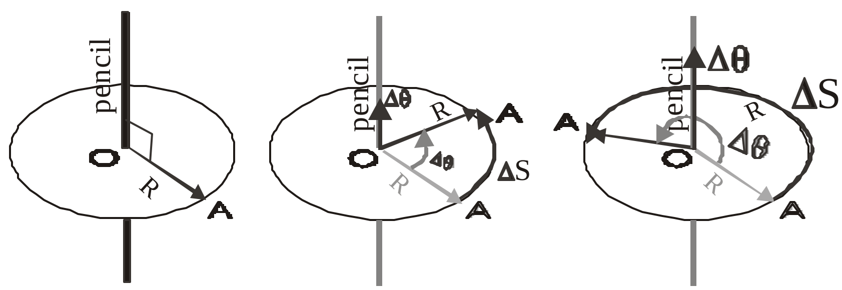
$v = \Delta S/\Delta t = R\Delta\theta/\Delta t = R\omega$

$a = \Delta v/\Delta t = R\Delta\omega/\Delta t = Ra$

The area relationships of Section 6.1.2 also apply here. That is, $\Delta\omega$ = area under α vs t curve, and $\Delta\theta$ = area under the ω vs t graph.

6.8.1.1 Exercise: With a compass make a circle from cardboard . Make a line from the center to the circumference. Label the line OA.. Punch a hole in the center of the circle. Put a pencil or a straw through the hole, with the pencil or straw being perpendicular to the circle. Put a bit of glue where the pencil and the circle touch.

As you twist the pencil the circle turns. The figure shows that as the line OA sweeps out larger angles $\Delta\theta$, the circumferential segment ΔS grows. Note that your action causing the rotation is on the axis perpendicular to the circle. So, the pencil is the axis of rotation. The rotation **Dq** is a vector along this axis of rotation. Describe what you observe as you it rotates, rotates fast, rotates slowly. Describe the behavior of the vector **Dq** during these rotations.



6.9 Rotation in 3D - The Vector Product

Rotational motion is represented in 3-Dimensions. This is because rotational motion is described through the vector product. Review Section 5.3.3.2. In the above exercise, 6.8.1.1, $\boldsymbol{\Delta\theta} = \mathbf{R}\mathbf{x}\boldsymbol{\Delta}\mathbf{S}/R^2$, the vector product of **R** and **ΔS**. The $1/R^2$ is to make the dimensions and the value correct. The vector $\boldsymbol{\Delta\theta}$ grows longer and longer as we turn the pencil more and more. For the vector, we use the Right Hand Rule. Stick the fingers of the Right Hand in the direction of the first vector **R**, curl them in the direction of **ΔS**, stick out your right thumb. It points in the direction of $\boldsymbol{\Delta\theta}$. Similarly , **w** = **R x v**/R^2 and **a** = **a x R**/R^2 . Conversely, **ΔS** = **Rx** $\boldsymbol{\Delta\theta}$.. **v** = **w x R** and **a** = **a x R** .The vectors **w** and **a** lie along the axis of rotation, while the vectors **R**, **v** and **a** lie in the plane of rotation.

These figures from Section 5.3.3.2 show the vector relationships for rotational motion.

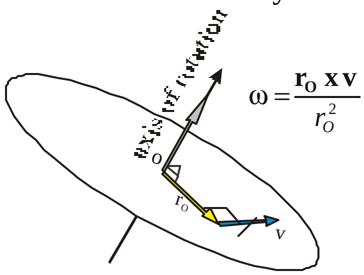


Figure 32

A point at position r_O from the axis of rotation, spinning about that axis with angular velocity ω, has linear velocity v at r_O as shown

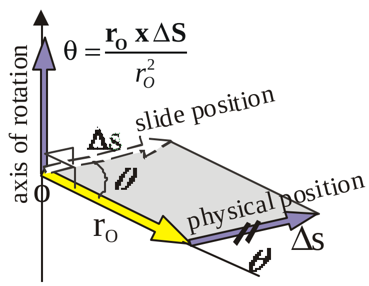


Figure 33

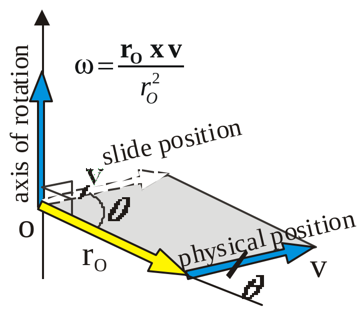


Figure 34

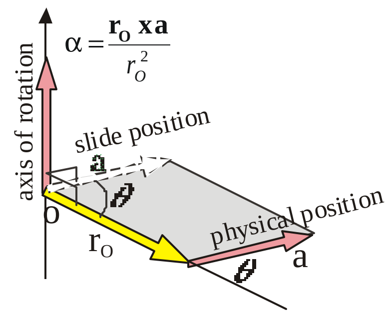


Figure 35

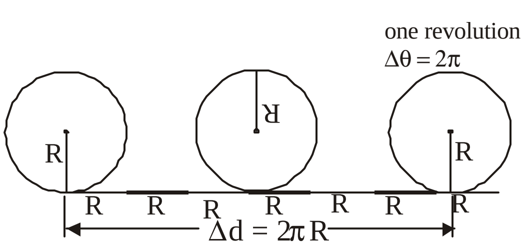
Angular displacement θ due to a linear displacement Δs at a position r_O from the axis of rotation.

Angular velocity ω due to a linear velocity v at a position r_O from the axis of rotation.

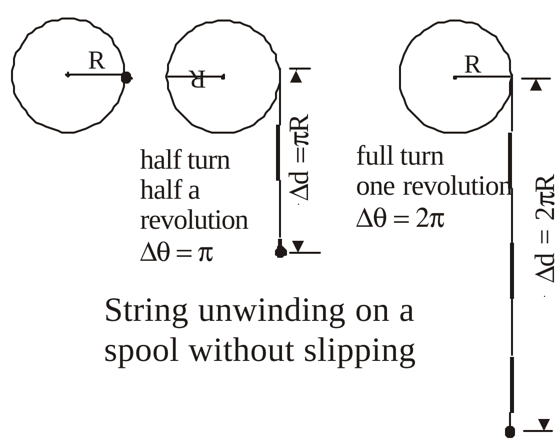
Angular acceleration α due to a linear acceleration a at a position r_O from the axis of rotation.

6.10 Rolling Without Slipping

Rolling without slipping is a very special case of rolling. It may be thought of as a type of rolling where the rotational motion and linear motion are uniquely mapped into each other. Each rotational displacement



Wheel rolling without slipping



String unwinding on a spool without slipping has a unique linear displacement, and vis-versa. The two figures show the situation for (1) a wheel rolling on a flat surface without slipping and (2) a string unwinding from a spool without slipping.

The relationship is **Δ d = R Dq ,**
linear displacement = Radius x angular displacement .

6.10.1 Exercise:

Take a round object like a jar top that has a diameter of at least 50mm (radius = ½diameter , D =2R). Put a mark on the rim . Use some moderately thick paper like the Sunday TV guide. Place the jar top on the paper with the mark on the rim touching the paper. Press down on the jar top as you roll the it in a straight line along the paper without slipping. Measure the diameter of the jar top and the length of the groove left in the paper. Do it again but allow the top to occasionally slip or skid on the paper.

6.11 Revolutions, Cycles, Frequency, Periods, and Radians

One revolution is one full turn of the circle. A full turn of the circle is 360^o or 2π radians (abbreviated rad) .

$$360 \text{ deg} = 2\pi \text{ rad} \qquad 1 \text{ rad} = 360/2\pi \text{ deg} = 180/\pi \text{ deg} = 57.2958 \text{ deg}$$

One revolution is also one cycle. Frequency, f, is the number of cycles completed in a second.. The period, T, is the time it takes to complete one cycle. So

$$f = 1/T \quad \text{and} \quad T = 1/f \quad .$$

A frequency of one cycle per second is one full circle a second, hence 1 cycle/sec = 2π rad /sec. So

$$\omega \text{ (rad/sec)} = 2\pi \text{ f} = 2\pi/T.$$

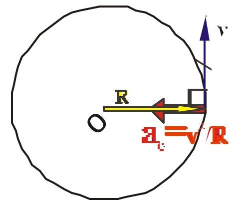
For example, 1 cycle/sec = 2π rad/sec (a period of 1 sec); 2 cycles/sec = 4π rad/sec (a period of ½ sec);
0.1 cycle/sec = 0.2π rad/sec (a period of 10 sec)

How about revolutions/min?

1 rev/min = 1 rev (2B) rad/rev /[~~min~~ (60)sec/~~min~~] = 2π/60 rad/sec = 0.10472 rad/sec.

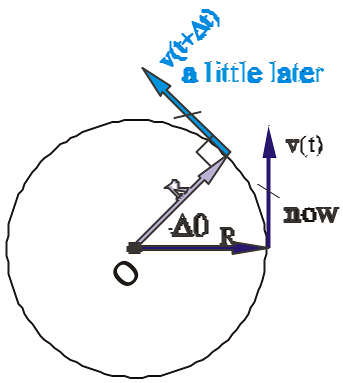
6.12 Uniform Circular Motion, Centripetal Acceleration and Circular Orbits

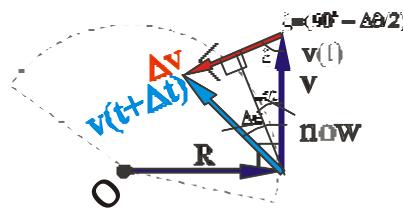
When an object travels in a circular path (around a circle with center O and radius R) with a constant speed, its velocity **v** is parallel to the circumference of the circle(tangent to the circle), hence perpendicular to the radius. It is called the tangential or orbital velocity. **This velocity has a constant magnitude ,** but **<u>its direction is changing</u>**. So there is an acceleration. This acceleration points to the center of the circle. It is called **centripetal acceleration** (<u>centri</u> - center, <u>petal</u> - seeking) and has the symbol $\mathbf{a_c}$.



6.12.1 Finding the centripetal acceleration, a_c

The figures show how $a_c = v^2/R$ comes about.





a. This Figure illustrates the situation with the tangential velocity at two different times, t and t+Δt.

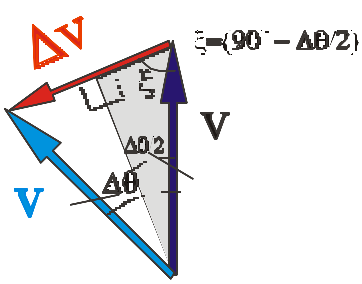
b. This shows the diagram for constructing the change in the velocity, **)v**.

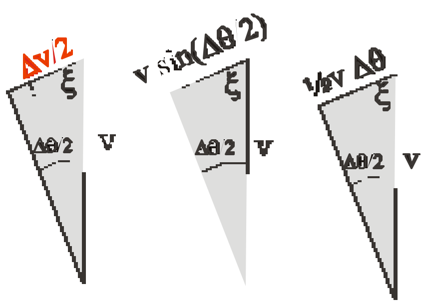
1. First slide the vector v(t+ Δt) so that its tail touches the tail of v(t). Δv follows from

$$v(t+\Delta t) = v(t) + \Delta v.$$

The base of this triangle is Δv and the opposite angle is $\Delta\theta$.

2. Since the magnitude of v is constant, the triangle is isosceles.





c. Let's focus on this isosceles triangle.

1. The perpendicular bisector cuts the triangle into two congruent right triangles. The base of each has length ½Δv with opposite angle $\Delta\theta/2$ and hypotenuse v.

2. By definition of the sine (sinθ=opp/hyp),

$$\Delta v/2 = v\sin(\Delta\theta/2).$$

3. Finally, the angle $\xi = 90^o - \Delta\theta/2$ because the sum of the angles of a plane triangle is 180°.

d. Finally, focusing on the highlighted right triangle and using the small angle approximation - Limit as θ->0 sinθ/θ->1(Section 4.3), We get that as $\Delta\theta$->0 , then

$$\Delta v/2 = 1/2\ v\Delta\theta.$$

Multiplying both sides by 2 and dividing both sides by Δt gives

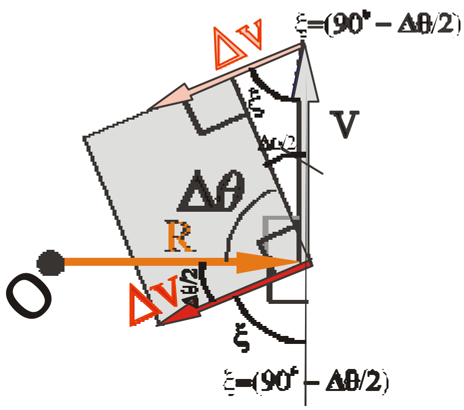
$$a = a_c = \Delta v/\Delta t = v\ \Delta\theta/\Delta t = v\,\omega.$$

But ω = v/R (See Rotational Motion Sec. 6.8.1), so

$$a = a_c = v\omega = v\ (v/R) = v^2/R,$$

or $$a = a_c = v\ \omega = (\omega R)\omega = \omega^2 R\ .$$

To discover the direction of the acceleration $\mathbf{a}_c$, let's focus on the angle between Δv and the radius vector R from the center O to the tail of v(t) "now".



e. Let's start by taking the figure in b, above, and slide Δv down to the tail of v(t) . Then focus on the gray rectangle.

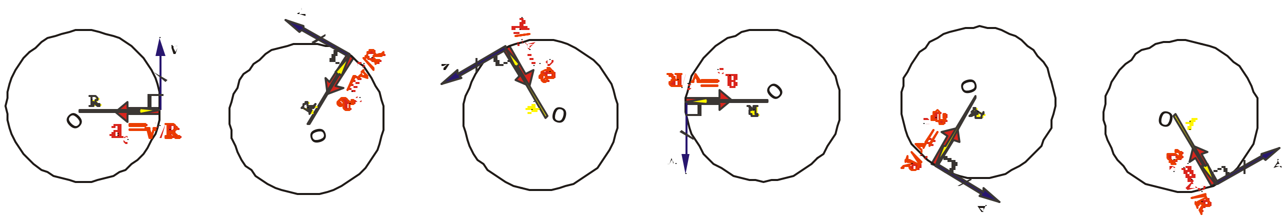
1. Now we see that Δv, hence a_c , simultaneously makes an angle $\Delta\theta/2$ to the negative direction of the vector R, and an angle $\xi = 90^o - \Delta\theta/2$ to the vector v.
2. So, as $\Delta\theta \to 0$, the angle between **a** and v becomes 90º (ξ becomes 90º), hence $\mathbf{a} = a_c$ is perpendicular to v and points towards the center of the circle. Hence,

$$\mathbf{a_c = v^2/R = \omega^2 R ;}$$

is perpendicular to v, anti-parallel to R, points to the center O.

6.12.2 v and a_c as it Moves around the Circle

As the object moves around the circle of radius R centered on O , the velocity, v, and the centripetal acceleration, a_c, maintain their relative relationship. The figure below illustrates this.



6.12.3 Centripetal Acceleration in terms of Between Tangential (Orbital) Velocity and Period

In circular motion there are two variables that are often observed directly. They are tangential velocity, v, and period, T (the time to make one complete cycle or revolution). In a car or airplane, the speedometer records the instantaneous value of v. For a turning wheel or in planetary motion, like the moon traveling around the earth, we observe the period , T. a_c in terms of v is already known. Now let's get a_c in terms of T.

The relation between v and T may be obtained from the fact that the distance traveled, Δd, in one period, T, is the circumference of the circle (= 2πR). So, time is

$$\Delta t = T \text{ (Period)}$$

and the distance traveled is

$$\Delta d = 2 \pi R \text{ (Circumference)}$$

By definition of v (= $\Delta d/\Delta t$ at constant speed), we get

$$v = \Delta d/\Delta t = \text{circumference/period} = 2\pi R/T .$$

So, in terms of the period T, a_c becomes;

$$a_c = v^2/R = (2\pi/T)^2 R = 4\pi^2 R/T^2 .$$

Table - a_c and T for various values of v and R

Centripetal Acceleration in m/s^2 (Gray Row) and Period of Rotation in sec (White Row) for various values of tangential velocity v (m/s) and radii R (meters) For clarity the units are in the first cell only.

tangential		Radius R (meters)					
velocity							
v (m/s)	**0.01 m**	**0.1**	**1**	**10**	**100**	**1000**	**10000**
0.01 m/s	1.0E-02 m/s^2	1.00E-03	1.00E-04	1.00E-05	1.00E-06	1.00E-07	1.00E-08
	6.28E+0 sec	*6.28E+01*	*6.28E+02*	*6.28E+03*	*6.28E+04*	*6.28E+05*	*6.28E+06*
0.1	1.00E+00	1.00E-01	1.00E-02	1.00E-03	1.00E-04	1.00E-05	1.00E-06
	6.28E-01	*6.28E+00*	*6.28E+01*	*6.28E+02*	*6.28E+03*	*6.28E+04*	*6.28E+05*
1	1.00E+02	1.00E+01	1.00E+00	1.00E-01	1.00E-02	1.00E-03	1.00E-04
	6.28E-02	*6.28E-01*	*6.28E+00*	*6.28E+01*	*6.28E+02*	*6.28E+03*	*6.28E+04*
10	1.00E+04	1.00E+03	1.00E+02	1.00E+01	1.00E+00	1.00E-01	1.00E-02
	6.28E-03	*6.28E-02*	*6.28E-01*	*6.28E+00*	*6.28E+01*	*6.28E+02*	*6.28E+03*
100	1.00E+06	1.00E+05	1.00E+04	1.00E+03	1.00E+02	1.00E+01	1.00E+00
	6.28E-04	*6.28E-03*	*6.28E-02*	*6.28E-01*	*6.28E+00*	*6.28E+01*	*6.28E+02*
1000	1.00E+08	1.00E+07	1.00E+06	1.00E+05	1.00E+04	1.00E+03	1.00E+02
	6.28E-05	*6.28E-04*	*6.28E-03*	*6.28E-02*	*6.28E-01*	*6.28E+00*	*6.28E+01*
10000	1.00E+10	1.00E+09	1.00E+08	1.00E+07	1.00E+06	1.00E+05	1.00E+04
	6.28E-06	*6.28E-05*	*6.28E-04*	*6.28E-03*	*6.28E-02*	*6.28E-01*	*6.28E+00*
100000	1.00E+12	1.00E+11	1.00E+10	1.00E+09	1.00E+08	1.00E+07	1.00E+06
	6.28E-07	*6.28E-06*	*6.28E-05*	*6.28E-04*	*6.28E-03*	*6.28E-02*	*6.28E-01*

6.12.2.1 Exercise:

By direct calculation verify all of the values in the above table. At minimum verify all values for a fixed value of v, say v = 10 m/s and plot graph of a_c vs R. Do the same for a fixed value of R, say R= 10 m, and plot a graph of a_c vs v. Do each verification in full with subscripts and units. For example,

$$a_c = \frac{v^2}{R} = \frac{[10m/s]^2}{10m} = \frac{[10^2][m^2/s^2]}{10m} = [\frac{10^2}{10}]*[\frac{m^2}{m}]/s^2 = 10m/s^2 \quad .$$

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