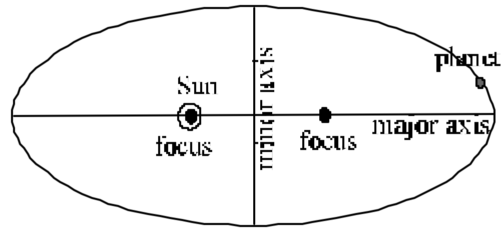
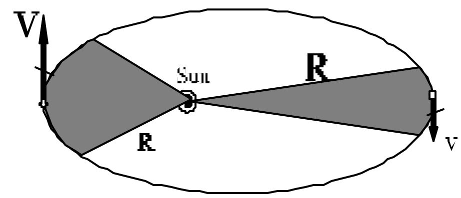
12 Gravity

12.1 Kepler's Laws of Planetary Motion



1. The planets travel around the sun in elliptical orbits with the sun at one of the foci (plural focus) of the ellipse.

2. As the planet travels in its orbit, it sweeps out the same area in a given amount of time.



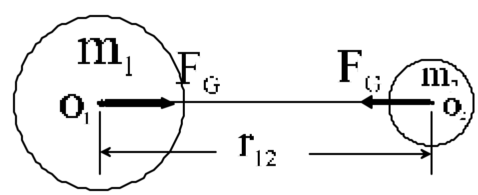
3. The square of period T of the planet's revolution around the sun is proportional to the cube of the mean radius R of the planet from the sun. That is, $T^2 = AR^3$ where A is a constant.

These are empirical laws. They catalogue discoveries from observations, but do not provide any reason for them. Such observations are critical to the development and verification of theories. A theory that does not predict the observed facts is considered invalid. A theory that does predict the observed facts is not necessarily true, but has a chance. These laws of Kepler are one of the main legs on which Newton's mechanics rests because Newton's mechanics successfully predicts them.

12.2 Newton's Law of Gravitation

Two masses, m_1 and m_2 with there centers separated by a distance r_{12} attract each other by a gravitational force , F_G , given by Newton's Law of Gravitation as

$$\mathbf{F}_G = -(Gm_1m_2 / r^2_{12})\, \mathbf{r^*} ,$$



where $\mathbf{r}^*$ is a unit vector along r_{12} , and G is the universal gravitational constant,

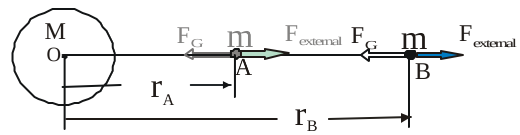
$$G = 6.67 \times 10^{-11}\ Nm^2/kg^2 \quad .$$

The force acts along the line joining their centers, O_1 and O_2, as shown. The minus sign says that it is an attractive force, it pulls the bodies towards each other.

12.3 Gravitational Potential Energy U

The gravity is a conservative force. The work done by an external force on a test mass m while moving it from point A to point B is

$$W_{A\to B} = -\int_{r_A}^{r_B} G\frac{mM}{r^2}dr = G\frac{mM}{r}\Big]_{r_A}^{r_B} = G\frac{mM}{r_B} - G\frac{mM}{r_A} = GmM\left(\frac{1}{r_B} - \frac{1}{r_A}\right)$$



Now, letting r_B go to infinity yields an absolute gravitational potential energy U(r) as

$$U(r_A) \equiv \lim_{r_B \to \infty} W_{A\to B} = GmM(\frac{1}{r_B \to \infty} - \frac{1}{r_A}) = GmM(0 - \frac{1}{r_A}) = -\frac{GmM}{r_A} \ .$$

So, in general, the gravitational potential energy E_P = U(r) is

$$E_P(r) = U(r) = - GmM/r$$

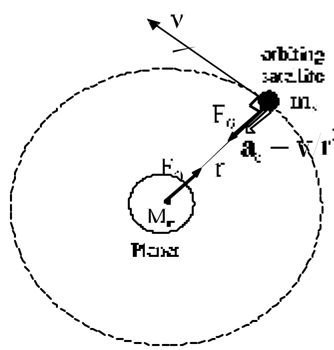
The negative value means that it is attractive. See section 12.5 for the meaning and full implication of this. This is an absolute potential energy, unlike the local expression in Section 10.3.2.1

12.3.1 **Exercise**: Show that on the surface of a “planet” of radius R and mass M that the local Gravitational potential energy of a mass m a height H above the surface is $E_{P\ Gravity}$ = U(R+H) - U(R) = (G M/R^2) H if H <<<R.

Hint:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + ... = \sum_{n=0}^{\infty} (-1)^n x^n$$

12.4 Satellites in Circular Orbits



When a satellite travels in uniform circular motion (See Section 6.12) around a planet, the satellite has a centripetal acceleration. The gravitation force between the planet and the satellite keeps satellite in the circular path, hence causing this .(See Chapter 9.) By Newton's 2nd Law of Motion we have

$$F = ma = ma_c = mv^2_{orbit} / r_{orbit}$$

But by Newton's Law of Gravitation, F = F_G = GMm/r^2, so

$$GMm/r^2 = mv^2_{orbit} / r_{orbit} \ ,$$

canceling m and an r and a bit of algebra gives

$$v_{orbit} = \sqrt{\frac{GM}{r_{orbit}}}$$

The period T and the frequency f of the orbit comes from $v_{orbit} = \omega r = 2\pi f r = 2\pi r/T$. So the period of the orbit, the time it takes for a complete cycle is

$$T_{orbit} = 2\pi\sqrt{\frac{r^3_{orbit}}{GM}}$$

If we square this equation we get

$$T^2_{orbit} = \frac{4\pi^2}{GM} r^3_{orbit}$$

which is Kepler's 3rd Law - "The square of the period is proportional to the cube of orbital radius". Now, solving for the mass of the parent object, M, gives

$$M = \frac{4\pi^2}{GT^2_{orbit}} r^3_{orbit}$$

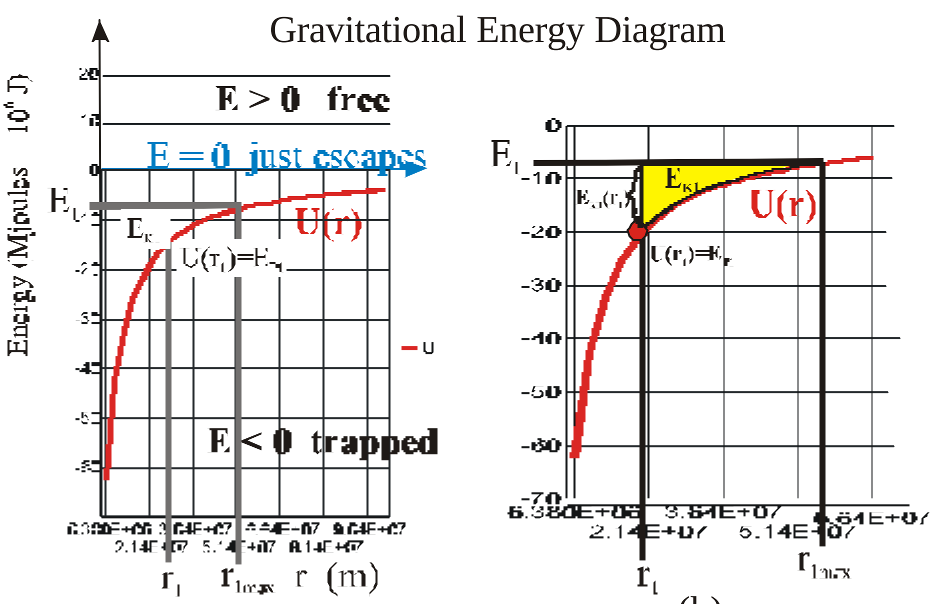
This can be called an **astronomical scale**. It yields the value of M from the measurements of the orbital radius r_{orbit} and period T_{orbit} of a satellite. From the time it takes the moon to travel around the earth and its orbital distance from earth, we find the earth's mass. Centuries ago that is how we found the masses of Jupiter, Saturn and the Sun.

12.4.1 Exercise:
Use Newton's Law of Gravitation and the relationship between weight W and mass m to show that the acceleration due to gravity g at the surface of a planet or star of radius R and mass M is

$$g = GM/R^2$$

12.5 Gravitational Energy Diagram

The diagrams here are called energy diagrams. They graph
1. vertical axis is the total energy, $E = E_P + E_K$ of a test mass m = 1kg
2. and its potential energy for U (r) = E_P on earth over the from r = earth's surface radius = 6.38×10^6 m to about 9.64×10^7m from the center of the earth.



(a) (b)

Figure (a) shows the general situation along with the particulars for the test mass with total energy $E_1 < 0$. Figure (b) shows a detailed of E_1 situation.

The total energy of the test mass is

$$E = E_K + E_P = ½ mv^2 - GMm/r . \quad (1)$$

This provides three situations of interest.

1. **E > 0 free**: Here ½ mv^2 > GMm/r . Hence as $r \to \infty$, $E \to \frac{1}{2}mv^2 > 0$

the body is totally free from the gravity of M with energy left over to keep moving.

2. **E=0 just escapes**: When ½ mv^2 = GMm/r , then E = 0. As . $r \to \infty$, $E \to \frac{1}{2}mv^2 = 0$

So, m has just enough E_K to escape the gravity of M as it comes to a stop; no energy left over. This value of v is called the escape velocity v_{escape}, give as

$$v_{escape} = \sqrt{\frac{2GM}{r}} \qquad (2)$$

3. **E < 0 trapped**: When ½ mv^2 < GMm/r , then E < 0, so the body m is trapped. The distance between the line E and the curve U(r) is the kinetic energy , $E_K(r)$, at r. That is,

$$E_K(r) = E - U(r).$$

If the total energy E <0 the line E hits U(r) at r_{max}. At this point the E_K=0 : The body stops and must fall back towards the earth. Figure (b) shows this in detail for when the test mass m has total energy $E_1 < 0$. $E_{K1}(r_1)$ and $U(r_1)$ are shown at $r = r_1$. The shaded region labeled E_{K1} shows the kinetic energy of m between r_1 and r_{max}.

SOME NUMBERS - Calculated densities, g, and escape velocities for bodies in our Solar System

Calculated values average densities, g and escape velocities for bodies in our Solar System

G (N m^2/kg^2)=	6.6726E-11						
	M	R	Avg Density	g	g/g(Earth)	v(escape)	v(escape)/
	(kg)	(m)	(kg/m^3)	(m/s^2)		(m/s)	(escapeEarth
Sun	1.990E+30	6.960E+08	1.409E+03	274.11	28.01	4.37E+05	55.28
Earth	5.970E+24	6.380E+06	5.488E+03	9.79	1.00	7.90E+03	1.00
moon	7.350E+22	1.740E+06	3.331E+03	1.62	0.17	1.68E+03	0.21
Mercury	3.280E+23	2.570E+06	4.613E+03	3.31	0.34	2.92E+03	0.37
Venus	4.820E+24	6.310E+06	4.580E+03	8.08	0.83	7.14E+03	0.90
Mars	6.420E+23	3.430E+06	3.798E+03	3.64	0.37	3.53E+03	0.45
Jupiter	1.890E+27	7.180E+07	1.219E+03	24.46	2.50	4.19E+04	5.30
Saturn	5.69E+26	6.030E+07	6.195E+02	10.44	1.07	2.51E+04	3.18

12.6.1 Exercise: By direct calculation verify all the calculated values in the above table. The calculated values assume that all bodies are perfectly uniform spheres. They are not quite.

Calculated values of Gravitational Forces between Sun and planets, average orbital velocities, v_{orbit}, periods T_{orbit}, and centripetal accelerations a_c of planetary orbits.

G (N m^2/kg^2)=	6.6726E-11					
M Sun (kg) =	1.990E+30					
	Mercury	Venus	Earth	Mars	Jupiter	Saturn
Mass (kg)	3.280E+23	4.820E+24	5.970E+24	6.420E+23	1.890E+27	5.69E+26
R orbit (m)	5.79E+10	1.08E+11	1.49E+11	2.28E+11	7.78E+11	1.43E+12
F Gravity (N)	1.30E+22	5.49E+22	3.57E+22	1.64E+21	4.15E+23	3.69E+22
v orbit (m/s)	4.79E+04	3.51E+04	2.99E+04	2.41E+04	1.31E+04	9.64E+03
T orbit (s)	7.60E+06	1.94E+07	3.14E+07	5.94E+07	3.74E+08	9.32E+08
T orbit (days)	87.925	223.990	362.971	687.060	4330.737	10791.863
T orbit (years)	0.242	0.617	1.000	1.893	11.931	29.732
a centripetal(m/s^2	3.96E-02	1.14E-02	5.98E-03	2.55E-03	2.19E-04	6.49E-05

12.6.2 Exercise: By direct calculation verify all the calculated values in the above table. The calculated values assume that all orbits are circles. They are not quite.

Some calculated values for Earth PE = U(r), orbital velocities v, periods T, and centripetal accelerations for various orbits around the Earth					
G (N m^2/kg^2)=	6.6726E-11	M Earth (kg)=	5.970E+24	m =	1.00 ' kg
	at the surface	spy satellite	geosychon-ous satellite	the moon	
R (m)	6.380E+06	6.65E+06	4.22E+07	3.83E+08	
PE (J)	-6.244E+07	-5.99E+07	-9.43E+06	-1.04E+06	
v orbit (m/s)	7.902E+03	7.74E+03	3.07E+03	1.02E+03	
a centripetal(m/s^2)	9.787E+00	9.00E+00	2.23E-01	2.72E-03	
T orbit (s)	5.073E+03	5400	86400	2.36E+06	
T orbit (min)	8.455E+01	90	1440	3.93E+04	
T orbit (hr)	1.409E+00	1.5	24	655.2	
T orbit (days)	5.872E-02	0.0625	1	27.3	

12.6.3 Exercise: By direct calculation verify all the calculated values in the above table. The calculated values assume that all orbits are circles and that the Earth is a uniform sphere.. None are quite true. The PE is for a 1 kg mass.

12.7 The Black Hole

A black hole is a stellar body with gravity strong enough on its surface that light cannot escape, hence its escape velocity, v_{escape} >= the speed of light, $c = 3 \times 10^8$ m/s . When we take

$$v_{escape} = c = \sqrt{\frac{2GM}{R_S}}$$

we get the Schwarzchild Radius, R_S for the radius of the black hole. Hence R_S becomes

$$R_S = \frac{2GM}{c^2} = 1.4828x10^{-27} M$$

Let M_S = mass of the sun = 1.99×10^{30} kg.

- o For $M = 1\ M_S$, then $R_S = 2.95 \times 10^3$ m. As a black hole the sun would have a radius of 2.95 km.
- o For $M = 100\ M_S$, then $R_S = 2.95 \times 10^5$ m.
- o For $M = 1000 M_S$, then $R_S = 2.95 \times 10^6$ m.
- o For $R_S = 6.96 \times 10^8$ m = radius of the sun, the mass of the black hole is $M = 2.36 \times 10^5\ M_S = 2.36 \times 10^5$ mass of the sun.

13. Elasticity

13.1 Hooke's Law and Elasticity

When a stress is applied to a body made of a non-rigid material, that body deforms. The deformation is called a strain. If the body is made of an elastic material, then the stress and strain are linearly related by **Hooke's Law**, as

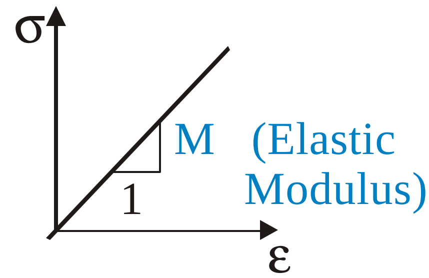
Stress = Elastic Modulus x Strain , (1)

where

Stress σ = Force/Area = F / A (2)
Units N/m^2 , p/in^2 .

and

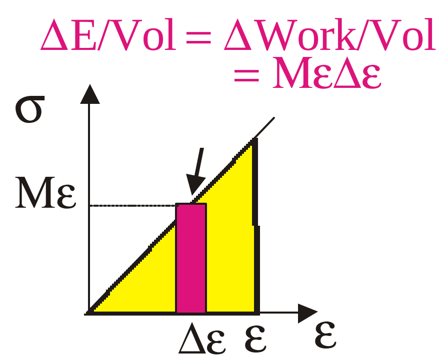
Strain ε = change in length / original length
= $\Delta L / L_o$. (3)



σ vs ε curve

The value of the elastic modulus depends on the type of stress and the material.

Stress (σ) = Modulus x Strain (ε) σ = M ε (4)



Elastic Energy

13.2 Elastic Energy/unit volume = ½ M ε^2 = ½ $\varepsilon\,\sigma^2$ = ½ σ^2/M, the area under the σ vs ε curve - the area of a triangle. It is the work per unit volume done by an external caused stress in causing the strain. Units J/m^3

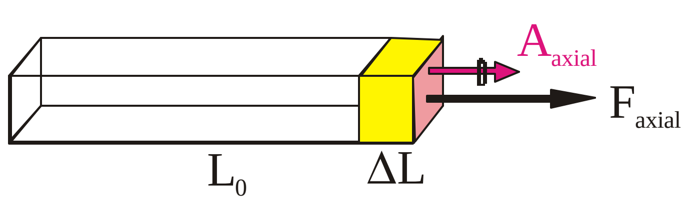
13.2.1 Conversion N/m^2 to p/in^2

$$1\frac{N}{m^2}=\frac{N(\,)\frac{kg}{N}(\,)\frac{p}{kg}}{[m(\,)\frac{cm}{m}(\,)\frac{in}{cm}]^2}=\frac{N(\frac{1}{9.81})\frac{kg}{N}(2.206)\frac{p}{kg}}{[m(100)\frac{cm}{m}(\frac{1}{2.54})\frac{in}{cm}]^2}=1.450E-04\frac{p}{in^2}$$

13.3 AXIAL - the body changes length due to axial stress

$F_{SHEAR}/A_{SHEAR} = \sigma_{AXIAL} = M_Y\ {}_A\varepsilon_{XIAL}$
$F_{AXIAL}/A_{AXIAL} = M_Y\ \Delta L/L_o$

The action is along (parallel to) the axis through the length of the body. It causes a change in length of the body.



M_Y is Young's Modulus

Lead 0.16x10^{11}N/m^2 , Aluminum 0.70x10^{11}N/m^2 , Steel 2.0 x10^{11}N/m^2 , Tungsten 3.6x10^{11}N/m^2

13.3.1 Behavior Under Axail Stress

	D	0.001 m	1 mm					
	Area =	7.85E-07 m^2		Vol =	7.85E-07 m^3			
	L=	1.0 m						

Force		σ		MY (N/m^2)	Delta L (m)	ε	PE/Vol (J/m^3)	PE J
1.0 N	0.1 kg	1.27E+06 N/m^2	Tungsten	3.60E+11	3.54E-06 m	3.54E-06	2.25E+00	1.77E-06
			Steel	2.00E+11	6.37E-06 m	6.37E-06	4.05E+00	3.18E-06
			Aluminum	7.00E+10	1.82E-05 m	1.82E-05	1.16E+01	9.09E-06
			Lead	1.60E+10	7.96E-05 m	7.96E-05	5.07E+01	3.98E-05
10.0 N	1.02 kg	1.27E+07 N/m^2	Tungsten	3.60E+11	3.54E-05 m	3.54E-05	2.25E+02	1.77E-04
			Steel	2.00E+11	6.37E-05 m	6.37E-05	4.05E+02	3.18E-04
			Aluminum	7.00E+10	1.82E-04 m	1.82E-04	1.16E+03	9.09E-04
			Lead	1.60E+10	7.96E-04 m	7.96E-04	5.07E+03	3.98E-03
100.0 N	10.2 kg	1.27E+08 N/m^2	Tungsten	3.60E+11	3.54E-04 m	3.54E-04	2.25E+04	1.77E-02
			Steel	2.00E+11	6.37E-04 m	6.37E-04	4.05E+04	3.18E-02
			Aluminum	7.00E+10	1.82E-03 m	1.82E-03	1.16E+05	9.09E-02
			Lead	1.60E+10	7.96E-03 m	7.96E-03	5.07E+05	3.98E-01

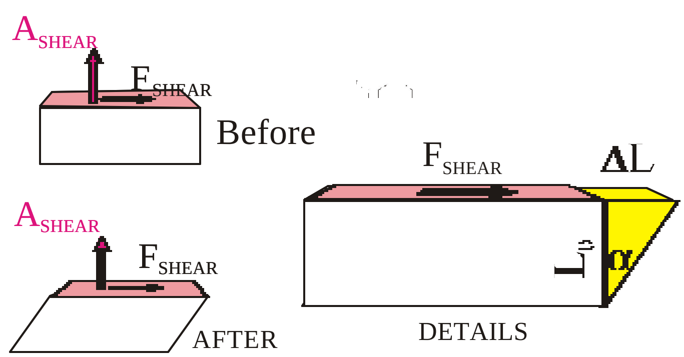
13.3.1 Exercise: By detailed calculation, including units, verify some of the values in this table.

13.4 SHEAR - the body changes shape due to shear stress

$F_{SHEAR}/A_{SHEAR} \quad \sigma_{SHEAR} = M_Y \ \varepsilon$

SHEAR

$F_{SHEAR}/A_{SHEAR} = M_S \ \Delta L/L_o = \tan\alpha$



The action is along and parallel to the outer surfaces of the body. It causes the body to change shape.

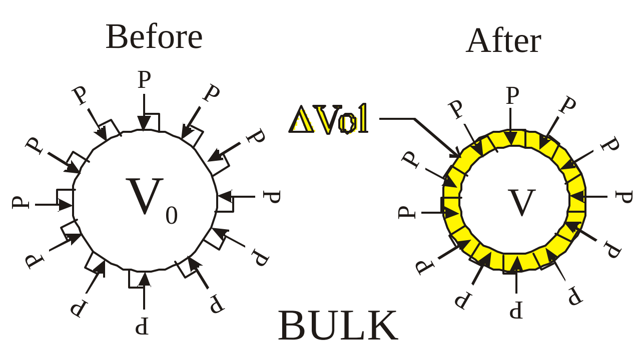
M_S is the shear Modulus

Lead $0.056\text{x}10^{11}$N/m^2 , Aluminum $0.30\text{x}10^{11}$N/m^2 , Steel $0.84 \text{ x}10^{11}$N/m^2 , Tungsten $1.5\text{x}10^{11}$N/m^2

13.5 BULK - the body changes volume due to pressure

Pressure = $F_{SHEAR}/A_{SHEAR\ BULK} = M_Y \ \varepsilon_{BULK}$

Pressure = $-M_B \ \Delta Vol/Vol_o$



The action, pressure, is always perpendicular to each point of the outer surface of the body. It causes a volume change of the body.

M_B is the bulk Modulus

Lead $0.077\text{x}10^{11}$N/m^2 , Aluminum $0.70\text{x}10^{11}$N/m^2 , Steel $1.6 \text{ x}10^{11}$N/m^2 , Tungsten $2.0\text{x}10^{11}$N/m^2
Water $0.022 \text{ x}10^{11}$N/m^2 , Ethyl alcohol $0.0091 \text{ x}10^{11}$N/m^2

The "-" sign is because ΔVol, the change in volume is always negative, the body becomes smaller when the pressure is positive.

13.5.1 Behavior Under Pressure

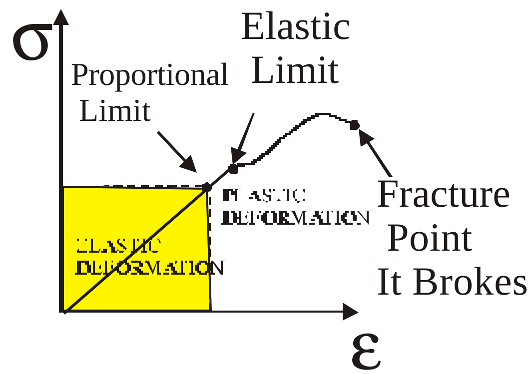
	R	1.0 m	Vol =	9.42E+0 m^3			
Pressure			MB (N/m^2)	Delta V (m^3)	ε	PE/Vol (J/m^3)	PE J
1.0E+04 N/m^2		Steel	1.60E+11	-5.89E-07	-6.25E-08	3.13E-04	2.95E-03
		Lead	7.70E+09	-1.22E-05	-1.30E-06	6.49E-03	6.12E-02
about 1 m deep in water		Water	2.20E+09	-4.28E-05	-4.55E-06	2.27E-02	2.14E-01
		Alcohol	9.10E+08	-1.04E-04	-1.10E-05	5.49E-02	5.18E-01
1.0E+06 N/m^2		Steel	1.60E+11	-5.89E-05	-6.25E-06	3.13E+00	2.95E+01
		Lead	7.70E+09	-1.22E-03	-1.30E-04	6.49E+01	6.12E+02
about 100 m deep in wate		Water	2.20E+09	-4.28E-03	-4.55E-04	2.27E+02	2.14E+03
		Alcohol	9.10E+08	-1.04E-02	-1.10E-03	5.49E+02	5.18E+03
1.0E+08 N/m^2		Steel	1.60E+11	-5.89E-07	-6.25E-08	3.13E-04	2.95E-03
		Lead	7.70E+09	-1.22E-05	-1.30E-06	6.49E-03	6.12E-02
about 10,000 m deep in water		Water	2.20E+09	-4.28E-05	-4.55E-06	2.27E-02	2.14E-01
		Alcohol	9.10E+08	-1.04E-04	-1.10E-05	5.49E-02	5.18E-01

13.5.1.1 **Exercise**: By detailed calculation, including units, verify some of the values in this table.

13.6 Breaking or Rupture Stress σ_{BREAK} is the stress at which the body of that material breaks.

For tension and shear the body breaks into two pieces. For compression and pressure the material crumbles.

Glass $10 \times 10^8 N/m^2$, Aluminum $2.2 \times 10^8 N/m^2$, Steel 5-20 $\times 10^8 N/m^2$, Pine Wood

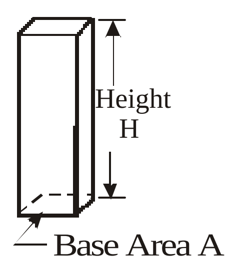


13.7 Exercise - Experience Stress

1. With both hands facing down, place one hand on top of the other. Press the top flat hand down with about a 5 p force.
2. Now ball you fist and press down on the back of your hand using the flat fist using same force as above.
3. Next , press down on the back of hand with one knuckle using the same force as above - if you can!

Describe what you feel in each case.
Note that if you do this with a sharp pencil or pen point, you would find the experience most penetration - literally.

13.8 Size Effects - limits on the size of bodies because they will collapse under their own weight.



Let's consider a rectangular body H high with base area A.

stress at base, σ_{BASE} = force on base / base area
= weight of body / base area
= weight density * Volume/base area

= ρg * Volume / base area
= ρg * Height * ~~base area~~ / ~~base area~~
= ρg * Height = ρ gH

At maximum height, H_{max} , the base crushes, that is F = F_{BREAK} , hence

$$H_{max} = \sigma_{BREAK} / \rho g \ .$$

For pine trees ρg = 50 p/f^3 and σ_{BREAK} = 300 p/in^2 . So

$$H_{max} = 300\ p/in^2/(50\ p/f^3) = (300/50)\ f^2 f/in^2 = 6\ f^2 f/in^2$$

using the conversion 12in = 1 f, or (12in/f) gives

$$= 6\ (12in/f)^2 f/in^2 = 5*144\ f = 720\ f.$$

So, if pine trees were built like this, on earth the tallest could be no more 720 f. Taller, it would crush its own base. This type of modeling provides estimates for the size limits on structures, plants, animals, and whatever else, arising from the mechanical strength of the materials. In reality, the tree would buckle long before reaching 720 f. Try to stand a flat sheet of paper on its edge. It collapses by bending. This is buckling. That is a calculation beyond this course.

	ρ	σ	Hmax		
	kg/m^3	N/m^2	m	f	12f storie
Steel	7800	1E+09	1.31E+04 m	4.29E+04 f	3573
Aluminum	2700	2.2E+08	8.31E+03 m	2.73E+04 f	2271
concrete	2000	3E+08	1.53E+04 m	5.02E+04 f	4181

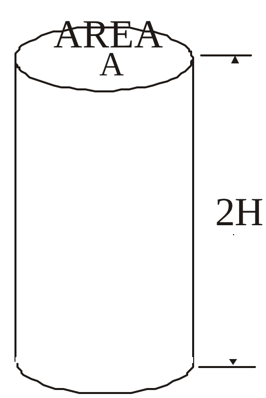
Very tall skyscrapers are quite possible.

13.9 The Butt Flop Revisited - more size effect.

On page 39 we discussed what happens when someone falls on their butt (rump, behind). We calculated the force as

$$F = -\ mgH/\Delta x$$

where H is half the height of the person and the Δx is the thickness of the fatty padding of the buttocks. Δx was taken to be fixed at about 1 cm = 10^{-2} m .



Let's look at things from a size and form point of view. A prototype "person" can be taken as shown in the figure. The figure retains it proportions as the size changes. Now stress σ = F/A is more important than the force itself, as you experienced in the last exercise - experiencing stress.

The mass m is

$$m = \rho Vol = \rho 2HA$$

where ρ is the mass density of the body. Then F becomes

$$F = mgH/\Delta x = g\rho 2HAH/\Delta x\ .$$

Dividing through by gives

$$\sigma = F/A = 2\rho g(H/\Delta x)H.$$

The factor 2ρg(H/Δx) is a constant - H and Δx change by the same factor as the size changes. So, the stress depends upon the height H. Small height, small stress, and visa versa.